ISRAEL JOURNAL OF MATHEMATICS, Vol. 73, No. 3, 1991

EMBEDDING FINITE GRAPHS INTO GRAPHS COLORED WITH INFINITELY MANY COLORS

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ABSTRACT

We prove that for any cardinal τ and for any finite graph H there is a graph G such that for any coloring of the pairs of vertices of G with τ colors there is always a copy of H which is an induced subgraph of G so that both the edges of the copy and the edges of the complement of the copy are monochromatic.

§0 Introduction

In this paper G, H denote graphs, i.e. sets of unordered pairs. First we define the "true embedding" partition symbol

$$G \rightarrowtail (H)^2_{\tau,\sigma}.$$

For graphs G, H and cardinals τ, σ the symbol is said to hold if for all colorings $f: G \to \tau$, $g: \bar{G} \to \sigma$, there are a $W \subset \bigcup G$, i.e. a set of vertices of G, and two colors $\mu < \tau$, $\nu < \sigma$ such that $G[W]$, the subgraph of G induced by W, is isomorphic to H and for $e \in [W]^2$, $e \in G$ implies $f(e) = \mu$ and $e \notin G$ implies $f(e) = \nu$. $G \not\rightarrowtail (H)^2_{\tau,\sigma}$, as usual, denotes the negation of this statement.

The main problem, if for a given triple H, τ and σ a suitable G exists at all, should be attributed to Deuber. The symbol, or rather a special instance of it, was introduced in [E–H–P], where we used the symbol $G \rightarrowtail (H)^2_\tau$ for the statement $G \rightarrowtail (H)^2_{\tau,1}$.

The first results were obtained in the early 1970's independently in [D], [E–H–P] and [N–R], all of which implied that

$$(1) \qquad \forall |H| < \omega \forall \tau < \omega \exists G \qquad G \rightarrowtail (H)^2_{\tau,\tau}.$$

†Research supported by Hungarian National Science Foundation OTKA grant 1805.
Received October 11, 1989

However in [E–H–P] we made attempts to generalize (1) for the case of infinite graphs. We proved

(2) $$\forall |H| \leq \omega \forall \tau < \omega \exists G \qquad G \rightarrowtail (H)^2_{\tau,\tau}.$$

For a countable H we found a G of cardinality 2^ω, and we also proved that a countable G will not do in general:

(3) $$\forall |G| \leq \omega \qquad G \not\rightarrowtail (K_{\omega,\omega})^2_{2,1}.$$

Here $K_{\omega,\omega}$ is the countable by countable complete bipartite graph. It was already clear from the proof of these results that unlike Ramsey's theorem, (1) and (2) will not easily generalize for infinite H and τ. The real reason was only found in our paper [H–K] with Komjáth. Indeed,

(4) It is consistent with ZFC that

$$\exists |H| = \omega_1 \forall G \qquad G \not\rightarrowtail (H)^2_{2,1}.$$

This can be proved by adding one Cohen real, and using a very simple graph H invented by Shelah several years earlier.

Shortly after this Shelah [S] proved a result from the other direction:

(5) It is consistent with ZFC that

$$\forall H \forall \tau \exists G \qquad G \rightarrowtail (H)^2_{\tau,\tau}.$$

He proved that this generalizes for arbitrary relational structures in place of graphs. We do not formulate this generalization here precisely.

(1)...(5) leave the problem open if (2) can be generalized for infinite τ. We still do not know this for $|H| = \omega$, but we can prove in ZFC the following

THEOREM. $\forall |H| < \omega \forall \tau \exists G \quad G \rightarrowtail (H)^2_{\tau,\tau}.$

It seems to be clear that this admits a Shelah type generalization for finite relational structures H, but we do not go into that. The rest of the paper will be devoted to the technically rather complicated proof of the theorem. Unfortunately, the G we find is rather large. For $|H| = k < \omega$ our G is of size $\exp_4(\exp_k(\tau)^+) \leq \exp_{k+5}(\tau)$, but we did not bother to save one or two exponents.

§1. Notation. λ^+-Saturated graphs and good ideals

In what follows G denotes a graph on the vertex set κ. With some abuse of notation we will write $G = G_0$ and $\bar{G} = [\kappa]^2 \setminus G = G_1$. For $x < \kappa$ and $i < 2$, we set

$$G_i(x) = \{y < \kappa : \{x, y\} \in G_i\}.$$

For an infinite cardinal λ, let

$$H = H(\lambda, \kappa) = \{\epsilon : \epsilon \text{ is a function} \wedge |\epsilon| \leq \lambda \wedge \mathrm{Dom}(\epsilon) \subset \kappa \wedge \mathrm{Range}(\epsilon) \subset 2\}.$$

For $\epsilon \in H$ let

$$G_\epsilon = \{y < \kappa : \forall x \in \mathrm{Dom}(\epsilon)\, y \in G_{\epsilon(x)}(x)\}.$$

LEMMA 0. *Assume $\lambda \geq \omega$, $\kappa = 2^\lambda$. There exists a graph G on the vertex set κ such that for all $\epsilon \in H$, $|G_\epsilon| = \kappa$. In any such G there are pairwise disjoint sets $A_\alpha : \alpha < \kappa$ such that*

$$|G_\epsilon \cap A_\alpha| = \kappa \quad \text{holds for} \quad \alpha < \kappa, \quad \epsilon \in H.$$

PROOF. $\kappa^\lambda = \kappa$. ∎

Indeed G is just a λ^+-saturated model of a graph on κ, i.e. a model in which all types of size $\leq \lambda$ are of cardinality κ.

In what follows G and $\{A_\alpha : \alpha < \kappa\}$ are objects satisfying Lemma 0. They of course depend on the choice of λ.

Our first aim is to define ideals in $\mathbb{P}(\kappa)$, the set of all subsets of κ. Let

$$D = \{\langle \epsilon_\xi : \xi < \lambda^+ \rangle : \forall \xi, \eta < \lambda^+ \epsilon_\xi \in H \wedge \xi \neq \eta \Rightarrow \mathrm{Dom}(\epsilon_\xi) \cap \mathrm{Dom}(\epsilon_\eta) = \emptyset\}.$$

Definition of good and very good κ-ideals

(a) $J \subset \mathbb{P}(\kappa)$ is a good κ-ideal if $[\kappa]^{<\kappa} \subset J$, J is proper, λ^+ complete and for all $\langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D$ there is a $\xi < \lambda^+$ with $G_{\epsilon_\xi} \notin J$.

(b) For $J \subset \mathbb{P}(\kappa)$ define

$$\tilde{J} = \{X \subset \kappa : \exists \langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D\ \forall \xi < \lambda^+\ X \cap G_{\epsilon_\xi} \in J\}.$$

(c) $J \subset \mathbb{P}(\kappa)$ is a very good κ-ideal if J is a good κ-ideal and $\tilde{J} = J$, i.e. for $X \subset \kappa$

$$(\exists \langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D\ \forall \xi < \lambda^+\ X \cap G_{\epsilon_\xi} \in J) \Rightarrow X \in J.$$

LEMMA 1. (a) *$[\kappa]^{<\kappa}$ is a good κ-ideal.*

(b) *If J is a good κ-ideal then $J \subset \tilde{J}$, and $\tilde{J}$ is a very good κ-ideal, i.e. $\tilde{\tilde{J}} = \tilde{J}$.*

PROOF. (a) is trivial as $\mathrm{cf}(\kappa) > \lambda$.

(b) Assume now that J is a good κ-ideal. This implies, by definition, that $\kappa \notin \tilde{J}$ hence $\tilde{J}$ is proper.

Assume $X_\nu \in \tilde{J}$ and let $\langle \epsilon_\xi^\nu : \xi < \lambda^+ \rangle \in D$ establish this for $\nu < \lambda$. We can choose, by induction on $\eta < \lambda^+$, ordinals $\xi(\nu,\eta) < \lambda^+$ in such a way that the functions $\{\epsilon^\nu_{\xi(\nu,\eta)} : \nu < \lambda,\ \eta < \lambda^+\}$ have pairwise disjoint domains. This can be done, as for any set $|T| \le \lambda$ and for any $\nu < \lambda$, $|\{\xi < \lambda^+ : \mathrm{Dom}(\epsilon_\xi^\nu) \cap T \neq \emptyset\}| \le \lambda$. Let now $\epsilon_\eta = \bigcup \{\epsilon^\nu_{\xi(\nu,\eta)} : \nu < \lambda\}$. Then $\langle \epsilon_\eta : \eta < \lambda^+ \rangle \in D$ and for $\eta < \lambda^+$

$$\left(\bigcup_{\nu<\lambda} X_\nu\right) \cap G_{\epsilon_\eta} \subset \bigcup_{\nu<\lambda} (X_\nu \cap G_{\epsilon^\nu_{\xi(\nu,\eta)}}) \in J, \qquad \text{hence } \bigcup_{\nu<\lambda} X_\nu \in \tilde{J}.$$

This proves that $\tilde{J}$ is λ^+-complete.

Assume now $X \in \tilde{\tilde{J}}$, i.e. there is an $\langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D$ such that for $\xi < \lambda^+$

$$X \cap G_{\epsilon_\xi} \in \tilde{J}.$$

Assume that $\langle \epsilon_\eta^\xi : \eta < \lambda^+ \rangle \in D$ establish this fact for $\xi < \lambda^+$. We can choose for $\zeta < \lambda$ ordinals $\xi(\zeta)$ and $\eta(\zeta) < \lambda^+$ in such a way that the functions $\{\epsilon_{\xi(\zeta)}, \epsilon^{\xi(\zeta)}_{\eta(\zeta)} : \zeta < \lambda^+\}$ have pairwise disjoint domains. Let $\bar{\epsilon}_\zeta = \epsilon_{\xi(\zeta)} \cup \epsilon^{\xi(\zeta)}_{\eta(\zeta)}$ for $\zeta < \lambda^+$. Then $\langle \bar{\epsilon}_\zeta : \zeta < \lambda^+ \rangle \in D$ and

$$X \cap G_{\bar{\epsilon}_\zeta} \subset X \cap G_{\bar{\epsilon}_{\xi(\zeta)}} \cap G_{\epsilon^{\xi(\zeta)}_{\eta(\zeta)}} \in J \qquad \text{for } \zeta < \lambda^+,$$

hence $X \in \tilde{J}$. Note that this fact implies $\forall \langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D\ \exists \xi < \lambda^+,\ G_{\epsilon_\xi} \notin \tilde{J}$ as well. ∎

COROLLARY 1. *Let $J_0 = \tilde{J}$ for $J = [\kappa]^{<\kappa}$. Then J_0 is a very good κ-ideal, and $A_\alpha \notin J_0$ for $\alpha < \kappa$.* ∎

We only need the existence of a J_0 satisfying the requirements of Corollary 1. In [E–H–P] we used a G and J_0 with $\lambda = \omega$, to prove (2). In the proof given here using J_0 we will have to define ideals in a direct product rather than in κ. This requires some preliminaries.

(1.1) First, for a $k < \omega$ we choose $\varphi_k = \varphi = \exp_\kappa(\tau)^+$, and $\lambda_k = \lambda = \exp_3(\varphi)$ $(\kappa_k = \kappa = 2^\lambda)$.

We set $A = \times_{\alpha<\varphi} A_\alpha$; A will be the underlying set of all the ideals, which will be called A-ideals. Set

$$P = \{B \subset A : B = \bigtimes_{\alpha<\varphi} B_\alpha, \forall \alpha < \varphi B_\alpha \subset A_\alpha\}, \text{ the set of "boxes".}$$

B, C, D will run over elements of P. For $B \in P$, B_α will denote the α-th projection. Similarly for $x \in A$, $x = \langle x_\alpha : \alpha < \varphi \rangle$ where $x_\alpha \in A_\alpha$.

Let

$$H^* = \{\epsilon : \epsilon = \langle \epsilon^\alpha : \alpha < \varphi \rangle \wedge \forall \alpha < \varphi\ \epsilon^\alpha \in H\}$$

and

$$D^* = \{\langle \epsilon_\xi : \xi < \lambda^+ \rangle : \forall \xi < \lambda^+ \epsilon_\xi \in H^* \wedge \forall \alpha < \varphi \langle \epsilon_\xi^\alpha : \xi < \lambda^+ \rangle \in D\}.$$

Note that in these definitions we do *not* require $\mathrm{Dom}(\epsilon^\alpha) \subset A_\alpha$.

For $\epsilon \in H^*$ we write $G_\epsilon = \mathsf{X}_{\alpha<\varphi} G_{\epsilon^\alpha} \cap A_\alpha \in P$. For a family $\mathcal{T} \subset \mathbb{P}(A)$, $\mathrm{gen}(\mathcal{T})$ is the λ^+-complete ideal generated by $\mathcal{T}$, i.e.

$$\mathrm{gen}(\mathcal{T}) = \{X \subset A : \exists \mathcal{T}' \subset \mathcal{T}\ |\mathcal{T}'| \leq \lambda \wedge X \subset \bigcup \mathcal{T}'\}.$$

Definition of good and very good A-ideals

(a) $I^0 = \mathrm{gen}\{B \subset A : \exists \alpha < \varphi B_\alpha \in J_0\}$.

(b) $I \subset \mathbb{P}(A)$ is a good A-ideal if $I^0 \subset I$, I is proper, λ^+-complete and for all $\langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D^*$ there is a $\xi < \lambda^+$ with $G_{\epsilon_\xi} \notin I$.

(c) For $I \subset \mathbb{P}(A)$ define

$$\tilde{I} = \{X \subset A : \exists \langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D^*\ \forall \xi < \lambda^+\ X \cap G_{\epsilon_\xi} \in I\}.$$

(d) $I \subset \mathbb{P}(A)$ is a very good A-ideal if I is a good A-ideal and $\tilde{I} = I$, i.e. for $X \subset A$

$$\exists \langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D^*\ \forall \xi < \lambda^+\ X \cap G_{\epsilon_\xi} \in I \Rightarrow X \in I.$$

Now we need the analogue of Lemma 1.

Lemma 2. (a) *I^0 is a good A-ideal.*

(b) *If $I \subset \mathbb{P}(A)$ is a good A-ideal then $\tilde{I}$ is a very good A-ideal.*

The proof of Lemma 2 is completely analogous to the proof of Lemma 1. However, since it is a basic ingredient of our proof we will give it in some detail. With a new abuse of notation we will use set theoretic operations defined on elements of H^* as if defined coordinatewise, e.g. if $\epsilon_0, \epsilon_1 \in H^*$, $\epsilon_0 \cup \epsilon_1$ is the sequence $\langle \epsilon_0^\alpha \cup \epsilon_1^\alpha : \alpha < \varphi \rangle$, etc.

Proof of (a). Assume $\langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D^*$.

Let $\alpha < \varphi$. As $A_\alpha \notin J_0$ there is a $\xi(\alpha) < \lambda^+$ such that $G_{\epsilon_\xi^\alpha} \cap A_\alpha \notin J_0$ for $\xi > \xi(\alpha)$. There is a $\xi^* < \lambda^+$ such that $\xi(\alpha) < \xi^*$ for $\alpha < \varphi$. We claim that for $\xi^* < \xi < \lambda^+$, $G_{\epsilon_\xi} \cap A \notin I^0$. Assume $\mathcal{T} \subset \{B \subset A : \exists \alpha < \varphi B_\alpha \in J_0\}$, $|\mathcal{T}| \leq \lambda$. For $B \in \mathcal{T}$ let $\alpha(B) = \alpha$ be such that $B_\alpha \in J_0$. Let $x_\alpha \in G_{\epsilon_\xi^\alpha} \cap A_\alpha \setminus \bigcup \{B_{\alpha(B)} : B \in \mathcal{T} \wedge \alpha(B) = \alpha\}$ for $\alpha < \varphi$. Then $x = \langle x_\alpha : \alpha < \varphi \rangle \in G_{\epsilon_\xi} \setminus \bigcup \mathcal{T}$, hence $G_{\epsilon_\xi} \notin I^0$. ∎

PROOF OF (b). Again $\tilde{I}$ is a proper ideal by the assumption. Now to prove that $\tilde{I}$ is λ^+-complete and $\tilde{\tilde{I}} = \tilde{I}$ we can follow the proof of Lemma 1 using the following facts:

Assume $T_\alpha \subset \kappa \wedge |T_\alpha| \leq \lambda$ for $\alpha < \varphi$ and $\langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D^*$. Then

$$|\{\xi < \lambda^+ : \exists \alpha < \varphi \ \mathrm{Dom}(\epsilon_\xi^\alpha) \cap T_\alpha \neq \emptyset\}| \leq \lambda.$$

If $\{\epsilon_\nu : \nu < \lambda\}$ have pairwise disjoint domains, i.e. $\forall \alpha < \varphi \ \mathrm{Dom}(\epsilon_\nu^\alpha) \cap \mathrm{Dom}(\epsilon_\mu^\alpha) = \emptyset$ for $\nu \neq \mu < \lambda$ then $\bigcup_{\nu<\lambda} \epsilon_\nu = \langle \bigcup \{\epsilon_\nu^\alpha : \nu < \lambda\} : \alpha < \varphi \rangle \in H^*$. ■

COROLLARY 2. *Let $I_0 = \tilde{I}^0$. Then I_0 is a very good A-ideal.* ■

§2. Further notation for τ-colored G's. Motivation

As in our theorem both G and $\bar{G}$ are colored with τ colors, we may assume that we are given an $f : [\kappa]^2 \to \tau$. First for $x \in \kappa$, $i < 2$ and $\nu < \tau$ we define

$$G_{i,\nu}(x) = \{y \in G_i(x) : f(\{x,y\}) = \nu\}.$$

Clearly $G_i(x) = \bigcup \{G_{i,\nu}(x) : \nu < \tau\}$. Define

$$T = \{\langle \alpha, \Delta_0, \Delta_1 \rangle : \{\alpha\} \cup \Delta_0 \cup \Delta_1 \subset \varphi \wedge \alpha \notin \Delta_0 \cup \Delta_1 \wedge \Delta_0 \cap \Delta_1 = \emptyset\}.$$

The elements of T will be called triples. Define

$$N_{\Delta_0,\Delta_1} = \{\langle g_0, g_1 \rangle : \forall i < 2 \ g_i \in {}^{\Delta_i}\tau\}.$$

The elements of N_{Δ_0,Δ_1} will be called Δ_0, Δ_1-patterns or, briefly, patterns. Note that $|N_{\Delta_0,\Delta_1}| \leq \tau^{|\Delta_0 \cup \Delta_1|} \leq 2^\varphi$.

(2.1) Let $\langle \alpha, \Delta_0, \Delta_1 \rangle \in T$, $\langle g_0, g_1 \rangle \in N_{\Delta_0,\Delta_1}$, $B \subset A$, $x \in B_\alpha$; we define $G^B_{\Delta_0,\Delta_1}(x) \subset B$ and $G^B_{g_0,g_1}(x) \subset B$ as follows:

$$G^B_{\Delta_0,\Delta_1}(x) = \bigtimes_{i<2} \bigtimes_{\beta \in \Delta_i} (G_i(x) \cap B_\beta) \times \bigtimes_{\beta \notin \Delta_0 \cup \Delta_1} B_\beta \qquad \text{and}$$

$$G^B_{g_0,g_1}(x) = \bigtimes_{i<2} \bigtimes_{\beta \in \Delta_i} (G_{i,g_i(\beta)}(x) \cap B_\beta) \times \bigtimes_{\beta \notin \Delta_0 \cup \Delta_1} B_\beta.$$

Note that both sets defined above are boxes.

The following is a basic fact:

LEMMA 3. *For $\langle \alpha, \Delta_0, \Delta_1 \rangle \in T$, $B \subset A$, $x \in B_\alpha$*

$$G^B_{\Delta_0,\Delta_1}(x) = \bigcup \{G^B_{g_0,g_1}(x) : \langle g_0, g_1 \rangle \in N_{\Delta_0,\Delta_1}\}.$$

PROOF. Using $G_i(x) \cap B_\beta = \bigcup \{G_{i,\nu}(x) \cap B_\beta : \nu < \tau\}$, this is just the distributive law. ■

We now formulate an obvious

COROLLARY 3. *Assume I is a very good A-ideal, $B \notin I$ and $\langle \alpha, \Delta_0, \Delta_1 \rangle$ is a triple. Then*

$$|\{x \in B_\alpha : G^B_{\Delta_0,\Delta_1}(x) \in I\}| \leq \lambda.$$

PROOF. For $x \in B_\alpha$ define $\epsilon^\beta_x \in H$. $\epsilon^\beta_x = \emptyset$ for $\beta \notin \Delta_0 \cup \Delta_1$ and for $i < 2$, $x \in \Delta_i$, $\mathrm{Dom}(\epsilon^\beta_x) = \{x\}$, $\epsilon^\beta_x(x) = i$. Then $G_{\epsilon^\beta_x} \cap B_\beta = G_i(x) \cap B_\beta$ for $\beta \in \Delta_i$, and so for $\boldsymbol{\epsilon}_x = \langle \epsilon^\beta_x : \beta < \varphi \rangle$ we have

$$G^B_{\Delta_0,\Delta_1}(x) = G_{\boldsymbol{\epsilon}_x} \cap B.$$

Since for a one-to-one sequence $\{x_\xi : \xi < \lambda^+\}$ we have $\langle \boldsymbol{\epsilon}_{x_\xi} : \xi < \lambda^+ \rangle \in D^*$, this proves the claim. ∎

Now our aim is to prove that, given a graph H with vertex set $\{y_0, \ldots, y_{k-1}\}$, we can find colors $\nu_0, \nu_1 < \tau$, $\alpha_0 < \cdots < \alpha_{k-1} < \varphi$, and $x_j \in A_{\alpha_j}$ such that $G[\{x_j : j < k\}]$ is isomorphic to H and $G[\{x_j : j < k\}]$, $\bar{G}[\{x_j : j < k\}]$ are monochromatic in colors ν_i, $i < 2$, respectively. Loosely speaking, for this we need two colors ν_0, ν_1, a set $\Gamma = \{\alpha_0, \ldots, \alpha_{k-1}\}$ such that for any large B, for any $j < k - 1$ and for any partition $\Delta_0 \cup \Delta_1 = \{\alpha_{j+1}, \ldots, \alpha_{k-1}\}$ the set $G^B_{\bar{\nu}_0\bar{\nu}_1}(x)$ is still large for some $x \in B_{\alpha_j}$, where $\bar{\nu}_i$ is the constant ν_i function on Δ_i. Unfortunately "large" will depend on the stage j we are in. Hence we will have to extend the very good A-ideal I_0 to very good A-ideals $I_0 \subset \cdots \subset I_{k-1}$, and define somehow which colors are good for these ideals. The main tool for this is given in the next chapter.

§3. Building a partition tree in the product set A

LEMMA 4. *Assume I is a very good A-ideal. $B \subset A$ $(B \in P)$ and $B \notin I$. Assume further that for every triple $\langle \alpha, \Delta_0, \Delta_1 \rangle \in T$ a set of patterns $S^0_{\alpha,\Delta_0,\Delta_1} \subset N_{\Delta_0,\Delta_1}$ is given in such a way that for $x \in B_\alpha$ and $\langle g_0, g_1 \rangle \notin S^0_{\alpha,\Delta_0,\Delta_1}$*

$$G^B_{g_0,g_1}(x) \in I.$$

Then there exist a $C \subset B$ $(C \in P)$, $I \subset J$ and $S_{\alpha,\Delta_0,\Delta_1} \subset S^0_{\alpha\Delta_0,\Delta_1}$ such that J is a very good A-ideal, $C \notin J$ and

$$\forall \langle \alpha, \Delta_0, \Delta_1 \rangle \in T\ \forall \langle g_0, g_1 \rangle \notin S_{\alpha,\Delta_0,\Delta_1}\ \forall x \in C_\alpha\ G^C_{g_0,g_1}(x) \in I, \tag{1}$$

$$\forall (D \subset C \wedge D \notin J)\ \forall \langle \alpha, \Delta_0, \Delta_1 \rangle \in T\ \forall \langle g_0, g_1 \rangle \in S_{\alpha,\Delta_0,\Delta_1}\ \exists x \in D_\alpha \text{ such that } G^D_{g_0,g_1}(x) \notin I. \tag{2}$$

PROOF. We start with a remark. $B \notin I$ implies that there are sets $Z_\alpha \subset \kappa$, $|Z_\alpha| \leq \lambda$ for $\alpha < \varphi$ such that for every $\epsilon \in H^*$ with $\forall \alpha < \varphi(\mathrm{Dom}(\epsilon^\alpha) \cap Z_\alpha = \emptyset)$ $G_\epsilon \cap B \notin I$. Denoting by H^*_B the set $\{\epsilon \in H^* : \forall \alpha < \varphi\ \mathrm{Dom}(\epsilon^\alpha) \cap Z_\alpha = \emptyset\}$, it is clear that for $\forall \langle \epsilon_\xi : \xi < \lambda^+ \rangle \in D^*\ \exists \xi_0 < \lambda^+\ \forall \xi_0 < \xi < \lambda^+$

$$\epsilon_\xi \in H^*_B.$$

We also remark that it is sufficient to prove the existence of a good A-ideal J as above with $C \notin \tilde{J}$. Indeed, if J satisfies the above requirements so does $\tilde{J}$ and, by Lemma 2, $\tilde{J}$ is a very good A-ideal.

Now we will define a generalized partition tree consisting of subsets of B. More precisely, for every $\psi \in {}^\nu\lambda$, $\nu \leq (2^{2^\varphi})^+$ and $\langle \alpha, \Delta_0, \Delta_1 \rangle \in T$ we define a subset $B^\psi \subset B$, $S^\psi_{\alpha,\Delta_0,\Delta_1} \subset S^0_{\alpha,\Delta_0,\Delta_1}$ and $\epsilon_\nu \in H^*_B$. For $\nu = 0$, $\psi = \emptyset$, $B^\emptyset = B$, $S^\emptyset_{\alpha,\Delta_0,\Delta_1} = S^0_{\alpha,\Delta_0,\Delta_1}$, $\epsilon^\emptyset = \langle \emptyset : \alpha < \varphi \rangle$.

Assume $0 < \nu \leq (2^{2^\varphi})^+$ and for $\psi \in {}^\mu\lambda$, $\mu < \nu$ we have defined all these functions in such a way that the following conditions (i)...(v) hold.

(i) $B^\psi \in P$ or $B^\psi \in I$, $\epsilon_\mu \in H^*_B$.

(ii) For $\psi' \subset \psi$, $B^{\psi'} \supset B^\psi$, $S^{\psi'}_{\alpha,\Delta_0,\Delta_1} \supset S^\psi_{\alpha,\Delta_0,\Delta_1}$.

(iii) For $\mu' < \mu$, $\epsilon_{\mu'} \subset \epsilon_\mu$ (i.e. $\forall \alpha < \varphi\ \epsilon^\alpha_{\mu'} \subset \epsilon^\alpha_\mu$).

(iv) $B \cap G_{\epsilon_\mu} \subset \bigcup \{B^\psi : \psi \in {}^\mu\lambda\}$.

(v) For $\langle g_0, g_1 \rangle \notin S^\psi_{\alpha,\Delta_0,\Delta_1}$ and for $x \in B^\psi_\alpha$

$$G^{B^\psi}_{g_0,g_1}(x) \in I.$$

In case ν is limit set $B^\psi = \bigcap_{\mu<\nu} B^{\psi|\mu}$, $\epsilon_\nu = \bigcup_{\mu<\nu} \epsilon_\mu$ and $S^\psi_{\alpha,\Delta_0,\Delta_1} = \bigcap_{\mu<\nu} S^{\psi|\mu}_{\alpha,\Delta_0,\Delta_1}$. It is left to the reader to check that (i) $\cdots$ (v) still hold. Assume now that $\nu = \mu + 1$ is a successor. Note that in this case $|\nu| \leq 2^{2^\varphi}$.

Let $(\psi, \rho) = \psi \cup \{\langle \mu, \rho \rangle\}$ for $\psi \in {}^\mu\lambda$. (ψ, ρ) is the general element of ${}^\nu\lambda$.

Let

$$K^\psi = \{D \subset B^\psi : \exists \langle \alpha, \Delta_0, \Delta_1 \rangle \in T\ \exists \langle g_0, g_1 \rangle \in S^\psi_{\alpha,\Delta_0,\Delta_1}\ \forall x \in D_\alpha\ G^D_{g_0,g_1}(x) \in I\}.$$

We claim that if $J = \mathrm{gen}(K^\psi \cup I)$ is a good A-ideal with $B^\psi \notin \tilde{J}$, then $B^\psi = C$, and $S^\psi_{\alpha,\Delta_0,\Delta_1} = S_{\alpha,\Delta_0,\Delta_1}$ prove the Lemma. Indeed (1) holds by (v) and (2) holds as if for some $D \subset B^\psi (D \in P)$ and for $\langle \alpha, \Delta_0, \Delta_1 \rangle \in T$, $\langle g_0, g_1 \rangle \in S_{\alpha,\Delta_0,\Delta_1}$ $\forall x \in D_\alpha$ $(G^D_{g_0,g_1}(x) \in I)$ holds, then $D \in J$, by the definition of K^ψ.

Hence from now on we assume that $\mathrm{gen}(K^\psi \cup I)$ is not a good A-ideal on B^ψ, i.e. for all $\psi \in {}^\mu\lambda$,

$$\exists \langle \epsilon^\psi_\xi : \xi < \lambda^+ \rangle \in D^*\ \forall \xi < \lambda^+\ (G_{\epsilon^\psi_\xi} \cap B^\psi \in \mathrm{gen}(K^\psi \cup I)) \tag{3.1}$$

holds, and we will arrive at a contradiction. First, using $|{}^{\mu}\lambda| \le \exp_3(\tau)^{\exp_2(\tau)} = \lambda$ and the remark made at the start of the proof, we can choose $\xi(\psi) < \lambda^+$ in such a way that $\epsilon_{\xi(\psi)} \in H_B^*$ and the functions

$$\epsilon_\mu \quad \text{and} \quad \epsilon^{\psi}_{\xi(\psi)} : \psi \in {}^{\mu}\lambda$$

have pairwise disjoint domains (coordinatewise of course) and then we can set

$$\epsilon_\nu = \epsilon_\mu \cup \{\epsilon^{\psi}_{\xi(\psi)} : \psi \in {}^{\mu}\lambda\} \in H_B^*.$$

Now, by the indirect assumption (3.1), for each $\psi \in {}^{\mu}\lambda$ we can choose subsets $B^{\psi,\rho} \subset B^{\psi}$, $\rho < \lambda$ in such a way that $B^{\psi,\rho} \in P$ or $B^{\psi,\rho} \in I$,

$$G_{\epsilon_\nu} \cap B^{\psi} \subset G_{\epsilon^{\psi}_{\xi(\psi)}} \cap B^{\psi} \subset \bigcup\{B^{\psi,\rho} : \rho < \lambda\}.$$

Moreover, for $B^{\psi,\rho} \notin I$ for some $\langle \alpha, \Delta_0, \Delta_1\rangle \in T$ and $\langle g_0, g_1\rangle \in S^{\psi}_{\alpha,\Delta_0,\Delta_1}$ we have

$$\forall x \in B^{\psi,\rho}_{\alpha}\ (G^{B^{\psi,\rho}}_{g_0,g_1}(x) \in I).$$

Let

$$S^{\psi,\rho}_{\alpha,\Delta_0,\Delta_1} = S^{\psi}_{\alpha,\Delta_0,\Delta_1} \setminus \{\langle g_0, g_1\rangle \in S^{\psi}_{\alpha,\Delta_0,\Delta_1} : \forall x \in B^{\psi,\rho}_{\alpha} G^{B^{\psi,\rho}}_{g_0,g_1}(x) \in I\}.$$

We have defined all the necessary sets for $\nu = \mu + 1$ and again it is easy to check that (i)...(v) hold. Moreover, we know that $B^{\psi,\rho} \notin I$ implies that

$$S^{\psi,\rho}_{\alpha,\Delta_0,\Delta_1} \subsetneqq S^{\psi}_{\alpha,\Delta_0,\Delta_1}.$$

Let σ briefly denote the cardinal $(2^{2^{\varphi}})^+ \le \lambda = \exp_3(\varphi)$. First of all, $\epsilon_\sigma \in H_B^*$. We now claim that, for some $\psi \in {}^{\sigma}\lambda$, $B^{\psi|\nu+1} \notin I$ holds for all $\nu < \sigma$. Indeed, otherwise, by the remark made at the beginning of the proof, and by (iv), we have $B \cap G_{\epsilon_\sigma} \notin I$ and at the same time

$$B \cap G_{\epsilon_\sigma} \subset \bigcup\{B^{\psi} : \psi \in {}^{\sigma}\lambda\} \subset \bigcup_{\nu<\sigma} \bigcup\{B^{\psi|\nu+1} : \psi \in {}^{\sigma}\lambda \wedge B^{\psi|\nu+1} \in I\} \in I$$

by the λ^+-completeness of I, a contradiction.

We know that for this $\psi \in {}^{\sigma}\lambda$ for each $\nu < \sigma$ there is an $\langle \alpha, \Delta_0, \Delta_1\rangle \in T$ with $S^{\psi|\nu+1}_{\alpha,\Delta_0,\Delta_1} \subsetneqq S^{\psi|\nu}_{\alpha,\Delta_0,\Delta_1}$. Define a mapping $h : [\sigma]^2 \to T$ as follows. For $\mu < \nu < \sigma$, $h(\mu, \sigma) = \langle \alpha, \Delta_0, \Delta_1\rangle$ for a triple satisfying $S^{\psi|\nu}_{\alpha,\Delta_0,\Delta_1} \subsetneqq S^{\psi|\mu}_{\alpha,\Delta_0,\Delta_1}$. Note that $|T| \le 2^{\varphi}$. By the Erdős–Rado Theorem $(2^{2^{\varphi}})^+ \to ((2^{\varphi})^+)^2_{2^{\varphi}}$ there are a $\Gamma \subset \sigma$, $|\Gamma| = (2^{\varphi})^+$ and a triple $\langle \alpha, \Delta_0, \Delta_1\rangle \in T$ such that for $\mu < \nu$, $\mu, \nu \in \Gamma$, $h(\mu, \nu) = \langle \alpha, \Delta_0, \Delta_1\rangle$. But then $S^{\psi|\mu}_{\alpha,\Delta_0,\Delta_1} \supsetneqq S^{\psi|\nu}_{\alpha,\Delta_0,\Delta_1}$ for $\mu < \nu \in \Gamma$. Considering that $S^{\varnothing}_{\alpha,\Delta_0,\Delta_1} \subset N_{\Delta_0,\Delta_1}$ and $|N_{\Delta_0,\Delta_1}| \le 2^{\varphi}$, this is a contradiction. ■

We just add two remarks.

COROLLARY 4. *Assume I, J, C and $S_{\alpha,\Delta_0,\Delta_1}$ satisfy the requirements of Lemma 4. Then for $\langle \alpha, \Delta_0, \Delta_1 \rangle \in T$*

$$S_{\alpha,\Delta_0,\Delta_1} \neq \emptyset.$$

PROOF. By Corollary 3, $G^C_{\Delta_0,\Delta_1}(x) \notin I$ for some $x \in C_\alpha$. By Lemma 3, then $G^C_{g_0,g_1}(x) \notin I$ for some $\langle g_0, g_1 \rangle \in N_{\Delta_0,\Delta_1}$. By (1) of Lemma 4, then $\langle g_0, g_1 \rangle \in S_{\alpha,\Delta_0,\Delta_1}$. ∎

COROLLARY 5. *Under the above conditions, if $\langle \alpha, \Delta_0, \Delta_1 \rangle \in T$, $\langle g_0, g_1 \rangle \in S_{\alpha,\Delta_0,\Delta_1}$, and $\Delta'_i \subset \Delta_i$ for $i < 2$ then $\langle g_0 \mid \Delta'_0, g_1 \mid \Delta'_1 \rangle \in S_{\alpha,\Delta'_0,\Delta'_1}$.*

PROOF. For all $x \in C_\alpha$, $G^C_{g_0,g_1}(x) \subset G^C_{g_0|\Delta'_0, g_1|\Delta'_1}(x)$. By (2), there is an $x \in C_\alpha$ with $G^C_{g_0,g_1}(x) \notin I$, hence by (1) $\langle g_0 \mid \Delta'_0, g_1 \mid \Delta'_1 \rangle \in S_{\alpha,\Delta'_0,\Delta'_1}$. ∎

§4. End of the proof

Let I_0 be the very good A-ideal defined in Corollary 2, $A = B_0$. Let $S^0_{\alpha,\Delta_0,\Delta_1} = N_{\Delta_0,\Delta_1}$. We define the very good A-ideals I_j, and $S^j_{\alpha,\Delta_0,\Delta_1}$ by induction on $j < k$ as follows. If $I_j, B_j, S^j_{\alpha,\Delta_0,\Delta_1}$ satisfy the assumptions of Lemma 4 let $I_{j+1}, B_{j+1}, S^{j+1}_{\alpha,\Delta_0,\Delta_1}$ satisfy the requirements of this Lemma. Note that $I_0 \subset \cdots \subset I_{k-1}$, $B_0 \supset \cdots \supset B_{k-1}$, $S^0_{\alpha,\Delta_0,\Delta_1} \supset \cdots \supset S^{k-1}_{\alpha,\Delta_0,\Delta_1}$, $B^{k-1} \notin I_j$.

Let $T_k = \{\langle \alpha, \Delta_0, \Delta_1 \rangle \in T : |\{\alpha\} \cup \Delta_0 \cup \Delta_1| \leq k \wedge \alpha < \Delta_0 \cup \Delta_1\}$. For $\langle \alpha, \Delta_0, \Delta_1 \rangle$, $\langle \alpha' \Delta'_0, \Delta'_1 \rangle \in T_k$ write $\langle \alpha, \Delta_0, \Delta_1 \rangle \sim \langle \alpha', \Delta'_0, \Delta'_1 \rangle$ iff the monotone map π from $\Delta_0 \cup \Delta_1$ onto $\Delta'_0 \cup \Delta'_1$ sends Δ_i onto Δ'_i, and write

$$\langle \alpha, \Delta_0, \Delta_1 \rangle \sim^* \langle \alpha', \Delta'_0, \Delta'_1 \rangle \qquad \text{iff } \langle \alpha, \Delta_0, \Delta_1 \rangle \sim \langle \alpha', \Delta'_0, \Delta'_1 \rangle$$

and $S^j_{\alpha,\Delta_0,\Delta_1} = S^j_{\alpha',\Delta'_0,\Delta'_1}$ for $j < k$. Considering that $|N_{\Delta_0,\Delta_1}| \leq \tau$ for $|\Delta_0 \cup \Delta_1| \leq k$, each equivalence class of $\sim$ is split into at most 2^τ equivalence classes of $\sim^*$, hence by the Erdős–Rado Theorem

$$\varphi \to (\tau^+)^{\leq k}_{2^\tau}$$

there is a set $\Gamma \subset \varphi$, $\operatorname{typ}\Gamma = \tau^+$ such that for $\{\alpha\}, \Delta_0, \Delta_1, \{\alpha'\}, \Delta'_0, \Delta'_1 \subset \Gamma$, $\langle \alpha, \Delta_0, \Delta_1 \rangle, \langle \alpha', \Delta'_0, \Delta'_1 \rangle \in T_k$, $\langle \alpha, \Delta_0, \Delta_1 \rangle \sim \langle \alpha', \Delta'_0, \Delta'_1 \rangle$ we have

$$\langle \alpha, \Delta_0, \Delta_1 \rangle \sim^* \langle \alpha', \Delta'_0, \Delta'_1 \rangle. \tag{4.1}$$

For $\nu < \tau$ and $\Delta \subset \Gamma$ let $\nu \mid \Delta$ denote the constant function with value ν and domain Δ. Let $\alpha = \min \Gamma$. Let $\Gamma \setminus \{\alpha\} = \Gamma_0 \cup \Gamma_1$ be a partition of $\Gamma \setminus \{\alpha\}$ into the union of two disjoint subsets of type τ^+. By Corollary 4, $S_{\alpha,\Gamma_0,\Gamma_1} \neq \emptyset$. Let $\langle g_0, g_1 \rangle \in S_{\alpha,\Gamma_0,\Gamma_1}$. There are subsets $\Gamma_i' \subset \Gamma_i$ and ordinals $\nu_i < \tau$ such that $\operatorname{typ} \Gamma_i' = \tau^+$ and $g_i | \Gamma_i' = \nu_i$ for $i < 2$. By Corollary 5, this means

$$\langle \nu_0 | \Gamma_0', \nu_1 | \Gamma_1' \rangle \in S^{k-1}_{\alpha,\Gamma_0',\Gamma_1'}.$$

Using the homogeneity (4.1), it follows that for all $\langle \alpha', \Delta_0, \Delta_1 \rangle \in T_k$, $\{\alpha'\} \cup \Delta_0 \cup \Delta_1 \subset \Gamma$

$$\langle \nu_0 | \Delta_0, \nu_1 | \Delta_1 \rangle \in S^{k-1}_{\alpha',\Delta_0,\Delta_1}.$$

Let now $\alpha_0 < \cdots < \alpha_{l-1}$, $\alpha_j \in \Gamma$ for $j < l \leq k$. If follows easily by induction on l that for every graph H with vertex set $\{y_j : j < l\}$ and for every $C \subset B^{k-1}$ $C \notin I_l$, there are $x_j \in C_{\alpha_j}$, $j < l$ in such a way that the map $y_j \mapsto x_j$ is an isomorphism of H and the graphs $G_i[\{x_j : j < l\}]$ are monochromatic in the colors ν_i for $i < 2$.

Indeed let $\Delta_i = \{\alpha_j : 0 < j < l : \{y_0, y_j\} \in H_i\}$ for $i < 2$. There is an $x_0 \in C_{\alpha_0}$ with $G^C_{\nu_0|\Delta_0,\nu_1|\Delta_1}(x_0) \notin I_{l-1}$.

Applying the induction hypothesis for

$$C' = G^C_{\nu_0|\Delta_0,\nu_1|\Delta_1}(x_0), \quad l-1 \quad \text{and} \quad H[\{y_i : 0 < i < l\}]$$

the claim follows. ∎

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